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***Abstract*— In this paper we propose a Sliding Mode Controller for a class of scalar bilinear systems with delay in both the input and the state. Such a class is considered since it has shown to be suitable for modelling and control of a class of turbulent flow systems. The stability and robustness analysis for the reaching phase in the controlled system are Lyapunov-based. However, since the sliding dynamics is infinite dimensional and described by an integral equation, we show that the stability and robustness analysis is simplified by using Volterra operator theory.**

I. Introduction

Turbulent Flow Control is a fundamental problem in several areas of science and technology, and improvements to address such a problem can produce a very favourable effect in, for example, costs reduction, energy consumption, and environmental impact [3]. Unfortunately, in general, model based control techniques find several obstacles to be applied to the problem of Flow Control. One of the main difficulties is that the *par excellence* model for flow is the set of Navier-Stokes equations that is very complicated for simulation and control design [3]. On the other hand, when the model is very simple it is hard to represent adequately the behaviour of the physical flow. In [1] the authors say that *the remaining missing ingredient for turning flow control into a practical tool is control algorithms with provable performance guarantees*. Hence, adequate models (a trade-off between simplicity, efficiency, and accuracy) and algorithms for Flow Control are required.

In [7] a model for a flow system was proposed, such a model consists in a bilinear differential equation with delays in the input and in the state. An attractive feature of the model is that, according to the experimental results, with some few parameters the model reproduces the behaviour of the physical flow with a *good precision*. The justification for using such a kind of equations as models for flow systems was presented in [5]. We reproduce that reasoning as a motivational example in Section II.

For a particular case of the model introduced in [7], a sliding mode controller was proposed in [6]. That control technique was chosen due to the switching features of the actuators. A good experimental performance was obtained with such a controller[1]. Hence, it is worth to continue with the study of the class of bilinear delayed systems and to develop general schemes for analysis and control design. In this paper we design a Sliding Mode Controller for a subclass of such systems by following the idea proposed in [6]. Nonetheless, the result of this paper differs from [6] in the following points.

- The dynamics on the sliding surface is infinite dimensional and is described by an integral equation. In [6] the asymptotic stability of the sliding motion was analysed in the frequency domain. In this paper we propose, as one of the main contributions, to analyse the stability properties of such dynamics by considering it as a Volterra integral equation. This allows us to simplify the analysis and to give simple conditions to guarantee asymptotic stability of the solutions. Hence we avoid the necessity of making a frequency domain analysis to determine the stability properties of an infinite dimensional system.
- The analysis of the reaching phase is Lyapunov-based, this is important because it is not only useful to establish stability properties, but also robustness, whose analysis is performed applying Volterra operator theory.
- Although the systems considered in this paper and those in [6] are similar, the assumptions on the parameters are different. This allows us to enlarge the class of systems which can be considered for the application of the proposed methodology.

Paper organization: In Section III a brief description of the control problem is given. Some properties of the system's solutions are studied in Section IV. The design and analysis of the proposed controller are explained in Section V. A robustness analysis is given in Section VI. A numerical example is shown in Section VII. Some final remarks are stated in Section VIII.

Notation: $\mathbb{R}$ denotes the set of real numbers. For any $a \in \mathbb{R}$, $\mathbb{R}_{\geq a}$ denotes the set $\{x \in \mathbb{R} : x \geq a\}$, and analogously for $\mathbb{R}_{>a}$. For any $p \in \mathbb{R}_{\geq 1}$, $L^p(J)$ denotes the set of measurable functions $x : J \subset \mathbb{R} \to \mathbb{R}$ with finite norm $\|x\|_{L^p(J)} = \left(\int_J |x(s)|^p \, \mathrm{d}s\right)^{\frac{1}{p}}$, and $L^\infty(J)$ denotes the set of measurable functions with finite norm $\|x\|_{L^\infty(J)} = \operatorname{ess\,sup}_{t\in J} |x(t)|$.

II. Motivational example

In this section we reproduce the example given in [5] on how a bilinear delayed differential equation can be obtained as a model for a flow system.

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[1]A video with some experiments, reported in [6], can be seen at https://www.youtube.com/watch?v=b5NnAV2qeno.

A unidimensional approximation to the Navier-Stokes equations is the Burgers' equation given by

$$\frac{\partial \mathbf{v}(t,x)}{\partial t} + \mathbf{v}(t,x)\frac{\partial \mathbf{v}(t,x)}{\partial x} = \nu \frac{\partial^2 \mathbf{v}(t,x)}{\partial x^2}\,, \qquad (1)$$

where $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}$ is the flow velocity field, $x \in \mathbb{R}$ is the spatial coordinate, and $\nu \in \mathbb{R}_{\geq 0}$ is the kinematic viscosity. Assume that $x \in [0, F]$ for some $F \in \mathbb{R}_{>0}$. Suppose that $\mathbf{v}(t,x) = \bar{\mathbf{v}}(x - ct)$, i.e. the solution of (1) is a travelling wave with velocity $c \in \mathbb{R}_{\geq 0}$, it has been proven that (1) admits this kind of solutions [4]. A model approximation of (1) can be obtained by discretizing (1) in the spatial coordinate. Here, we use central finite differences, for the spatial derivatives, with a mesh of three points (and step of $h = F/2$). Thus

$$\begin{aligned}\frac{\partial \mathbf{v}(t,F/2)}{\partial t} + \tfrac{\mathbf{v}(t,F/2)}{F}\left[\mathbf{v}(t,F) - \mathbf{v}(t,0)\right] = \\ \tfrac{4\nu}{F^2}\left[\mathbf{v}(t,F) - 2\mathbf{v}(t,F/2) + \mathbf{v}(t,0)\right]. \qquad (2)\end{aligned}$$

Since $\mathbf{v}$ is assumed to be a travelling wave, it has a periodic pattern in space and time. In particular, note that $\mathbf{v}(t, F/2) = \bar{\mathbf{v}}(F/2 - ct) = \mathbf{v}(t + F/(2c), F) = \mathbf{v}(t - F/(2c), 0)$. Now, define $y(t) = \mathbf{v}(t, F)$ and $u(t) = \mathbf{v}(t, 0)$, thus, (2) can be rewritten as

$$\begin{aligned}\dot{y}(t) = -\tfrac{1}{F} y(t-\varsigma)u(t-2\varsigma) + \tfrac{1}{F} y(t)u(t-\varsigma) + \\ \tfrac{4\nu}{F^2}\left[y(t-\varsigma) - 2y(t) + u(t-\varsigma)\right],\end{aligned}$$

where $\varsigma = F/2c$. Hence, in [5], [7], [6] the authors propose a more general model for separated flow control:

$$\dot{y} = \sum_{i=1}^{N_1} a_i y(t-\tau_i) + \sum_{i=1}^{N_2}\left[\sum_{k=1}^{N_3} \bar{a}_k y(t-\bar{\tau}_k) + b_j\right] \times u(t-\varsigma_k)\,. \qquad (3)$$

Observe that this approximating model still recovers two main features of the original flow model: first, it is nonlinear; and second, it is infinite dimensional.

III. Problem Statement

Consider the system

$$\begin{aligned}\dot{x}(t) &= a_1 x(t-\tau_1) - a_2 x(t-\tau_2) \\ &\quad + \left[c_1 x(t-\bar{\tau}_1) - c_2 x(t-\bar{\tau}_2) + b\right] u(t-\varsigma)\,, \qquad (4)\end{aligned}$$

where $a_1, a_2, c_1, c_2, b, \tau_1, \tau_2, \bar{\tau}_1, \bar{\tau}_2 \in \mathbb{R}_{\geq 0}$. We assume that all the delays are bounded and constant. We also assume that the initial conditions of (4) are $x(t) = 0$ for all $t < 0$ and $x(0) = x_0$ for some $x_0 \geq 0$.

The control objective is to drive the state of the system to a constant reference $x^* \in \mathbb{R}_{>0}$. Such an objective must be achieved under the following general restrictions:

- Since the equation is used to model a positive physical system, some conditions on the model parameters have to be given to guarantee that the solutions of (4) can only take nonnegative values.
- Due to the physical nature of the on/off actuator, the control input is restricted to take values from the set $\{0, 1\}$.

IV. System's Properties

As stated in Section III we require some features of the solutions of (4) to guarantee that it constitutes a suitable model for the physical system. In this section we study the conditions on the parameters of (4) that guarantee non-negativeness and boundedness of the solutions. Of course, existence and uniqueness of solutions must be guaranteed. To this aim we rewrite (4) as

$$\begin{aligned}\dot{x} = a_1 x(t-\tau_1) + c_1 u(t-\varsigma)x(t-\bar{\tau}_1) - a_2 x(t-\tau_2) - \\ c_2 u(t-\varsigma)x(t-\bar{\tau}_2) + bu(t-\varsigma)\,, \qquad (5)\end{aligned}$$

that can be seen as a linear delayed system with time-varying coefficients. The term $bu(t-\varsigma)$ is considered as the input. In a first time, we will consider the general case $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and, then, restrict it to $u : \mathbb{R}_{\geq 0} \to \{0, 1\}$. A locally absolutely continuous function that satisfies (5), for almost all $t \in [0, \infty)$, and its initial conditions for all $t \leq 0$ is called a solution of (5) [2]. Hence, if in addition to the assumptions in the previous section we assume that $u : [0, \infty) \to \mathbb{R}$ is a Lebesgue-measurable locally essentially bounded function, then the solution of (5) exists and it is unique, see Appendix A. Such a definition of solution is adequate for the analysis made in this section, however, for the closed-loop behaviour analysis, we will also consider another framework, see Remark 1 in Section V.

A. Nonnegative solutions

We have said that the model has to be guaranteed to provide nonnegative solutions. Thus, we first search for some conditions that guarantee that the solutions of (5) are nonoscillatory[2]. Consider (5) and define $P(t) = a_1 + c_1 u(t-\varsigma)$ and $N(t) = a_2 + c_2 u(t-\varsigma)$.

Lemma 1 ([2], Corollary 3.13): Consider (5) with $b = 0$. If $\min(\tau_2, \bar{\tau}_2) \geq \max(\tau_1, \bar{\tau}_1)$, $N(t) \geq P(t)$ for all $t \geq t_0$, and there exists $\lambda \in (0, 1)$ such that

$$\limsup_{t\to\infty} \int_{t-\max(\tau_2,\bar{\tau}_2)}^{t-\min(\tau_1,\bar{\tau}_1)} \left(N(s) - \lambda P(s)\right) \mathrm{d}s < \frac{\ln(1/\lambda)}{e}\,,$$

$$\limsup_{t\to\infty} \int_{t-\max(\tau_2,\bar{\tau}_2)}^{t} \left(N(s) - \lambda P(s)\right) \mathrm{d}s < \frac{1}{e}\,,$$

then, the fundamental solution of (5) is such that $X(t, s) > 0$, $t \geq s \geq t_0$, and (5) has an eventually positive solution with an eventually nonpositive derivative.

Now, having nonoscillation conditions for (5) we can state the following.

Corollary 1: Consider (5) with $b \geq 0$. Suppose that the assumptions of Lemma 1 hold. Assume that $x(t) = 0$, $u(t) = 0$ for all $t < 0$ and $x(0) = x_0$ for some $x_0 \geq 0$. If $u(t) \geq 0$ for all $t \geq 0$, then $x(t) \geq 0$ for all $t \geq 0$.
The proof is straightforward through the solution representation by using the fundamental function, see Lemma 3.

[2]For the definition of a nonoscillatory solution see e.g. [11], [2].

Note that, in particular, the integral conditions of Lemma 1 are satisfied if

$$\left[a_2 - \frac{1}{e}(a_1 + c_1)\right] \max(\tau_2, \bar{\tau}_2) < \frac{1}{e}\,.$$

Although this is only sufficient, it constitutes a simple formula to verify the integral conditions of Lemma 1.

B. Boundedness of solutions

Observe that the nonoscillation conditions of Lemma 1 also guarantee the boundedness of the system's trajectories for $b = 0$. For the case $b \neq 0$ we have the following result.

Lemma 2: Consider (4) with its parameters satisfying Lemma 1, and with the initial conditions $x(t) = 0$, $u(t) = 0$ for all $t \leq 0$. If $b \neq 0$,

$$N(t) - P(t) \geq \alpha\,, \quad \forall\, t \geq 0\,, \tag{6}$$

for a strictly positive α, and $u(t) = 1$ $\forall t \geq 0$, then the solution of (5) is such that $x(t) \leq \bar{x}$ for all $t \geq 0$ and

$$\lim_{t\to\infty} x(t) = \bar{x}\,, \quad \bar{x} = \frac{b}{a_2 + c_2 - a_1 - c_1}\,. \tag{7}$$

Proof: According to Lemma 1, if $b = 0$, then we can ensure that there exists t_1 such that $x(t) > 0$ and $\dot{x}(t) \leq 0$ for all $t \geq t_1$. Hence, there exists $t_2 \geq t_1$ such that for all $t \geq t_2$

$$\begin{aligned}\dot{x} \leq\, & a_1 x(t - \max(\tau_1, \bar{\tau}_1)) + c_1 u(t - \varsigma) x(t - \max(\tau_1, \bar{\tau}_1)) \\ & - a_2 x(t - \min(\tau_2, \bar{\tau}_2)) - c_2 u(t - \varsigma) x(t - \min(\tau_2, \bar{\tau}_2)) \\ \leq\, & P(t) x(t - \max(\tau_1, \bar{\tau}_1)) - N(t) x(t - \min(\tau_2, \bar{\tau}_2))\,,\end{aligned}$$

thus, since $N(t) - P(t) \geq \alpha$, we can ensure that $\lim_{t\to\infty} x(t) = 0$, see e.g. [2, Theorem 3.4]. Now, for the particular case $u(t) = 1$ and $b = 0$, (4) is time-invariant and the asymptotic behaviour of $x(t)$ guarantees that $x = 0$ is asymptotically stable, therefore, it is exponentially stable and its fundamental solution $X(t, s)$ is exponentially bounded (see e.g. [11], [9]). Hence, for the case $b \neq 0$, $u(t) = 1$, the solution of (4) can be expressed as (see Lemma 3 in Appendix A)

$$x(t) = X(t, t_0) x(0) + \int_{t_0}^{t} X(t, s) b\, \mathrm{d}s\,.$$

Since $X(t, s)$ decreases exponentially in t, $x(t)$ is bounded, moreover, $x(t)$ increases monotonically due to the input term. Thus $\lim_{t\to\infty} x(t)$ exists and it is some constant $\bar{x}$, therefore, $\lim_{t\to\infty} \dot{x}(t) = 0 = -(a_2 + c_2 - a_1 - c_1)\bar{x} + b$. This equality gives the limit value (7). ■

V. Sliding Mode Controller

In this section we present the Sliding Mode Controller for (4), but first, define $k : \mathbb{R} \to \mathbb{R}$ given by

$$k(r) = k_{a_1}(r) - k_{a_2}(r) + k_{c_1}(r)\,, \tag{8}$$

where

$$\begin{aligned} k_{a_1}(r) &= \begin{cases} a_1, & r \in [\min(\varsigma, \tau_1), \max(\varsigma, \tau_1)], \\ 0, & r \notin [\min(\varsigma, \tau_1), \max(\varsigma, \tau_1)], \end{cases} \\ k_{a_2}(r) &= \begin{cases} a_2, & r \in [\varsigma, \tau_2], \\ 0, & r \notin [\varsigma, \tau_2], \end{cases} \\ k_{c_1}(r) &= \begin{cases} c_1, & r \in [\varsigma, \bar{\tau}_1], \\ 0, & r \notin [\varsigma, \bar{\tau}_1]. \end{cases} \end{aligned}$$

Theorem 1: If system (4) satisfies the conditions of Lemma 1, the condition (6), $\varsigma \leq \bar{\tau}_1$, and

$$\int\limits_{\min(\varsigma,\tau_1)}^{\tau_2} |k(r)|\, \mathrm{d}r < 1\,, \tag{9}$$

then, for any $x^* \in (0, \bar{x})$ (where $\bar{x}$ is given by (7)), the solution of the closed loop of (4) with the controller

$$u(t) = \tfrac{1}{2}(1 - \operatorname{sign}(\sigma_0(t) - \sigma^*))\,, \tag{10}$$

$$\begin{aligned}\sigma_0(t) = x(t) &+ a_1 \int_{t-\tau_1}^{t} x(s)\, \mathrm{d}s - a_2 \int_{t-\tau_2}^{t} x(s)\, \mathrm{d}s + \\ & c_1 \int_{t-\bar{\tau}_1+\varsigma}^{t} x(s)\, \mathrm{d}s + \int_{t-\varsigma}^{t} [c_1 x(s - \bar{\tau}_1 + \varsigma) - \\ & c_2 x(s - \bar{\tau}_2 + \varsigma) + b]\, u(s)\, \mathrm{d}s\,,\end{aligned} \tag{11}$$

where $\sigma^* = x^*\left[1 - a_2(\tau_2 - \varsigma) + a_1(\tau_1 - \varsigma) + c_1(\bar{\tau}_1 - \varsigma)\right]$, establishes a sliding motion in finite-time on the surface $\sigma_0(t) = \sigma^*$, and the sliding motion converges exponentially to x^*.

The design procedure is explained through the proof of the theorem given in the following sections. Note that for implementation, the following equivalent formula can also be used

$$\begin{aligned}\sigma_0(t) = x(t) + \int_0^t \{&(a_1 + c_1 - a_2)x(s) - a_1 x(s - \tau_1) + \\ & a_2 x(s - \tau_2) - c_1 x(s - \bar{\tau}_1 + \varsigma)(1 - u(s)) + \\ & [-c_2 x(s - \bar{\tau}_2 + \varsigma) + b] - [c_1 x(s - \bar{\tau}_1) - \\ & c_2 x(s - \bar{\tau}_2) + b]\, u(s - \varsigma)\}\, \mathrm{d}s\,.\end{aligned}$$

A. Sliding variable

Let us, from (10), define the sliding variable as $\sigma(t) = \sigma_0(t) - \sigma^*$. The time derivative of σ is

$$\begin{aligned}\dot{\sigma}(t) = &-(a_2 - a_1 - c_1)x(t) - c_1 x(t - \bar{\tau}_1 + \varsigma) + \\ & [c_1 x(t - \bar{\tau}_1 + \varsigma) - c_2 x(t - \bar{\tau}_2 + \varsigma) + b]\, u(t).\end{aligned} \tag{12}$$

Observe that σ_0 is acting as a kind of predictor since it allows us to have u without delay in (12). Now, let us verify that the trajectories of (4) in closed loop with (10) reach and remain on the sliding surface $\sigma = 0$ in finite-time. To this end, we substitute (10) in (12) to obtain the differential equation

$$\dot{\sigma}(t) = -\tfrac{1}{2} g_1(t) \operatorname{sign}(\sigma(t)) + g_2(t)\,, \tag{13}$$

where

$$\begin{aligned} g_1(t) &= c_1 x(t - \bar{\tau}_1 + \varsigma) - c_2 x(t - \bar{\tau}_2 + \varsigma) + b\,, \\ g_2(t) &= g_1(t)/2 + (a_1 + c_1 - a_2)x(t) - \\ & \qquad c_1 x(t - \bar{\tau}_1 + \varsigma)\,. \end{aligned} \tag{14}$$

Before we proceed to prove the establishment of a sliding regime on $\sigma = 0$, we have to guarantee the existence of solutions of (13).

Remark 1: Note that (13) can be seen as a nonautonomous differential equation with discontinuous right-hand side, therefore, we can use the definition of solutions given by Filippov in [8, p. 50][3]. But, g_1 and g_2 in (13) depend on x, and it is the solution of the functional differential equation (4) that in turn depends on σ through the input u. However, if we study recursively the system (4), (13), on the intervals $[n\varsigma, (n+1)\varsigma)$, $n = 0, 1, 2...$, we can see that the Filippov approach still works. Indeed, from the assumptions of the initial conditions for (4), $u(t) = 0$ for $t \in [0, \varsigma)$, hence, in such an interval the solutions of (4) do not depend on σ. Therefore, in the same interval, (13) can be seen as a simple differential equation with discontinuous right-hand side. Now, the solutions of (4) are not affected for the values of $\sigma(t)$ for any $t \in [\varsigma, 2\varsigma)$, thus, in such interval, (13) is a simple differential equation with discontinuous right-hand side, and so forth.

Consider the Lyapunov function candidate $V(\sigma) = \frac{1}{2}\sigma^2$, whose derivative along (13) is given by

$$\begin{aligned}\dot{V} &= \sigma(g_2(t) - \tfrac{1}{2} g_1(t)\operatorname{sign}(\sigma))\,, \\ &= -\tfrac{1}{2}(g_1(t) - 2g_2(t)\operatorname{sign}(\sigma))|\sigma|\,.\end{aligned}$$

Hence, V is a Lyapunov function for (13) if $g_1(t) - 2g_2(t)\operatorname{sign}(\sigma) \geq 0$. Let us start with the case $\sigma > 0$. In this case we have $g_1(t) - 2g_2(t) = (a_2 - a_1 - c_1)x(t) + c_1 x(t - \bar{\tau}_1 + \varsigma)$. Therefore, since the solutions of (4) are guaranteed to be nonnegative, $g_1(t) - 2g_2(t) \geq 0$. For the case $\sigma < 0$ we have $g_1(t) - 2g_2(t) = 2(b - (a_2 - a_1 - c_1)x(t) - c_2 x(t - \bar{\tau}_2 + \varsigma))$. Note that since $\sigma(0) < \sigma^*$ then $x(0) < \sigma^*$, and we know for this case that $x(t)$ is bounded from above by $\bar{x}$. This clearly implies that $b - (a_2 - a_1 - c_1)x(t) - c_2 x(t - \bar{\tau}_2 + \varsigma) \geq 0$.

Up to now, we have proven that $\sigma = 0$ is Lyapunov-stable, however, to guarantee finite-time convergence of $\sigma(t)$ to the origin, we have to verify that $g_1(t) - 2g_2(t)\operatorname{sign}(\sigma)$ is bounded from below by a strictly positive constant. The condition $\sigma > 0$ implies that $x_0 > \sigma^*$. If $x(t)$ is increasing it is convenient for the analysis, however, the critic situation is when $x(t)$ is decreasing and $x(t) < \bar{x}$. Note that, in such a case, $u = 0$ necessarily. Now, suppose that for some t_1 we have $x(t_1) = x^*$, then

$$\begin{aligned}\sigma(t_1) = -a_2 \int_{t-\tau_2}^{t} x(s)\,\mathrm{d}s + a_1 \int_{t-\tau_1}^{t} x(s)\,\mathrm{d}s + \\ c_1 \int_{t-\bar{\tau}_1+\varsigma}^{t} x(s)\,\mathrm{d}s - a_2(\tau_2 - \varsigma) + \\ a_1(\tau_1 - \varsigma) + c_1(\bar{\tau}_1 - \varsigma)\,,\end{aligned}$$

which is clearly negative. Hence, we can guarantee that, for the case $\sigma > 0$, $x(t)$ is bounded from below by x^*, and therefore, $g_1(t) - 2g_2(t) \geq (a_2 - a_1)x^*$.

[3]For the particular case of (13), the three methods given in [8, p. 50-56] to construct the differential inclusion coincide, see also [14].

Now, for practical purposes, let us define

$$\begin{aligned}S(t) = x(t) - a_2 \int_{t-\tau_2}^{t-\varsigma} x(s)\,\mathrm{d}s + \\ a_1 \int_{t-\tau_1}^{t-\varsigma} x(s)\,\mathrm{d}s + c_1 \int_{t-\bar{\tau}_1}^{t-\varsigma} x(s)\,\mathrm{d}s\,. \end{aligned} \tag{15}$$

Observe that the sliding variable σ can be rewritten as $\sigma(t) = S(t) - \sigma^* + R(t)$, where

$$\begin{aligned}R(t) = -\int_{t-\varsigma}^{t} \left[(a_2 - a_1 - c_1)x(s) + c_1 x(s - \bar{\tau}_1 + \varsigma)\right] \mathrm{d}s + \\ \int_{t-\varsigma}^{t} \left[c_1 x(s - \bar{\tau}_1 + \varsigma) - c_2 x(s - \bar{\tau}_2 + \varsigma) + b\right] u(s)\,\mathrm{d}s\,.\end{aligned}$$

Now, we want to prove that $b - (a_2 - a_1 - c_1)x(t) - c_2 x(t - \bar{\tau}_2 + \varsigma)$ is strictly positive when $\sigma < 0$. In this case the critic situation is when x is monotonically increasing. This happens only if $u = 1$. Note that in such situation

$$\begin{aligned}\sigma(t) = S(t) - \sigma^* + \int_{t-\varsigma}^{t} [b - (a_2 - a_1 - c_1)x(s) - \\ c_2 x(s - \bar{\tau}_2 + \varsigma)]\,\mathrm{d}s\,,\end{aligned}$$

where the term $\int_{t-\varsigma}^{t} [b - (a_2 - a_1 - c_1)x(s) - c_2 x(s - \bar{\tau}_2 + \varsigma)]\,\mathrm{d}s$ is strictly positive. Note also that $S(t) - \sigma^* \geq x(t) - x^*$. Hence, if for some t_1 we have that $x(t_1) = x^*$ then $\sigma(t_1) \geq 0$. Thus, we can conclude that $b - (a_2 - a_1 - c_1)x(t) - c_2 x(t - \bar{\tau}_2 + \varsigma)$ is bounded from below by $b - (a_2 + c_2 - a_1 - c_1)x^*$ when $\sigma < 0$. Therefore, we have proven that the sliding mode is established in finite-time.

B. Sliding dynamics

To obtain the dynamics on the sliding surface $\sigma = 0$, we use the Equivalent Control method [15], see also [16], [8], [14]. To compute the equivalent control, we make $\dot{\sigma}(t) = 0$ and obtain that

$$\begin{aligned}[c_1 x(t - \bar{\tau}_1 + \varsigma) - c_2 x(t - \bar{\tau}_2 + \varsigma) + b]\, u(t) = \\ -(a_1 + c_1 - a_2)x(t) - c_1 x(t - \bar{\tau}_1 + \varsigma)\,.\end{aligned}$$

By substituting this expression in the equation for $\sigma(t) = 0$ we obtain that the sliding dynamics is given by the integral equation

$$S(t) - \sigma^* = 0\,, \tag{16}$$

where S is given by (15). Hence, our objective is to prove that the solution $x(t)$ of (16) converges exponentially to x^*. Here we are going to use the results provided in Appendix B. First, let us rewrite (16) in a more suitable way. Define the change of variable $z(t) = x(t) - x^*$, thus, from the dynamics on the sliding surface, we obtain the following integral equation

$$\begin{aligned}z(t) - a_2 \int_{t-\tau_2}^{t-\varsigma} z(s)\,\mathrm{d}s + a_1 \int_{t-\tau_1}^{t-\varsigma} z(s)\,\mathrm{d}s + \\ c_1 \int_{t-\bar{\tau}_1}^{t-\varsigma} z(s)\,\mathrm{d}s = 0\,.\end{aligned}$$

Note that this equation can be rewritten as follows

$$z(t) + \int_{t^*}^{t} k(t-s)z(s)\,\mathrm{d}s = f(t)\,, \quad t \geq t^*\,, \tag{17}$$

where t^* is the reaching time to the sliding surface (i.e. the minimum t such that $\sigma(t) = 0$), k is given by (8) replacing the parameter r by $t-s$, and

$$f(t) = -\int_{t^*-\tau_2}^{t^*} k(t-s)\phi(s)\,\mathrm{d}s\,, \quad \phi(t) = z(t)\,, \forall t \leq t^*\,.$$

Observe that (17) is a Volterra integral equation of the second type and the kernel k of the integral is a convolution kernel. Now, we can state directly the following result.

Theorem 2: If $k : \mathbb{R}_{\geq t^*} \times \mathbb{R}_{\geq t^*} \to \mathbb{R}$ is a measurable kernel with $||k||_{L^p(\mathbb{R}_{\geq t^*})} < 1$, then for any $f \in L^1(\mathbb{R}_{\geq t^*})$ there exists a unique solution of (17) and it is such that $z \in L^1(\mathbb{R}_{\geq t^*})$. Moreover, $z(t) \to 0$ exponentially as $t \to \infty$.

Proof: First we claim that f is in $L^1(\mathbb{R}_{\geq t^*})$, see the verification in Appendix C. According to Lemma 5, the assumptions in Theorem 1 guarantee that $||k||_{L^p(\mathbb{R}_{\geq t^*})} < 1$. Thus, we can use Lemma 4 to guarantee existence and uniqueness of solutions of (17). Now, Lemmas 6 and 7 guarantee the exponential stability of $z = 0$. ■

Since $z = 0$ is exponentially stable, $x(t) \to x^*$ exponentially on the sliding surface.

VI. Robustness

In this section we analyse the robustness of the closed loop of (4) with (10), (11). For this, consider the system

$$\dot{x}(t) = a_1 x(t-\tau_1) - a_2 x(t-\tau_2) + [c_1 x(t-\bar{\tau}_1) - c_2 x(t-\bar{\tau}_2) + b]\, u(t-\varsigma) + \delta(t)\,, \tag{18}$$

where $\delta : \mathbb{R} \to \mathbb{R}$ is an external disturbance. We assume that $\|\delta\|_{L^\infty(\mathbb{R}_{\geq 0})} = \Delta$ for some finite $\Delta \in \mathbb{R}_{\geq 0}$. Considering (18), the time derivative of the sliding variable σ is

$$\dot{\sigma}(t) = -\tfrac{1}{2} g_1(t)\,\mathrm{sign}(\sigma(t)) + g_2(t) + \delta(t)\,, \tag{19}$$

where g_1 and g_2 are given by (14). Consider $V(\sigma) = \frac{1}{2}\sigma^2$ as a Lyapunov function candidate for (19). The derivative of V along (13) is given by

$$\dot{V} \leq -\tfrac{1}{2}\left[g_1(t) - 2g_2(t)\,\mathrm{sign}(\sigma) - |\delta(t)|\right]|\sigma|\,.$$

In Section V we proved that there exists a strictly positive ϵ such that $g_1(t) - 2g_2(t)\,\mathrm{sign}(\sigma) \geq \epsilon$ for all t along the reaching phase, thus if $\Delta < \epsilon$, then $\dot{V} \leq -\frac{1}{2}[\epsilon - \Delta|]\,|\sigma|$ and the sliding regime is established in finite-time. Nevertheless, since the sliding variable contains delayed terms of the control, the establishment of the sliding mode does not guarantee the complete disturbance rejection, see e.g. [12], [13]. Thus, let us analyse the behaviour of the sliding motion in the presence of the disturbance δ. By using again the equivalent control method (by taking into account the disturbance) we obtain the sliding dynamics $S(t) - \sigma^* - \delta(t) = 0$. If we

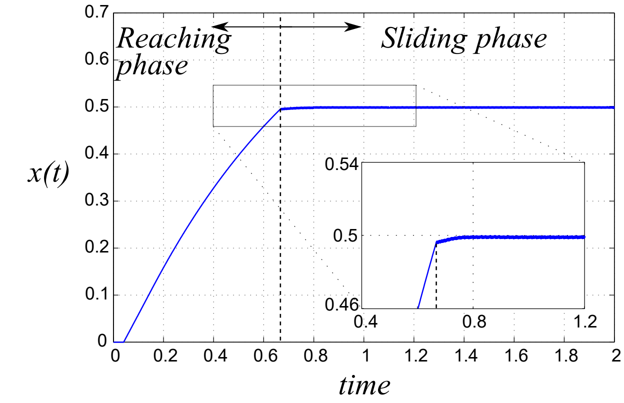

Fig. 1. State of the system in the nominal case.

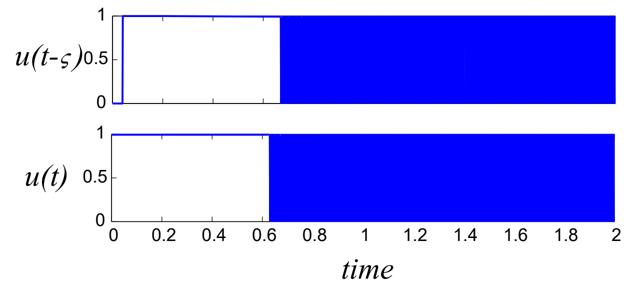

Fig. 2. Control signals in the nominal case.

use again the change of variable $z(t) = x(t) - x^*$, then the sliding dynamics can be rewritten as

$$z(t) + \int_{t^*}^{t} k(t-s)z(s)\,\mathrm{d}s = f(t) + \delta(t)\,, \quad t \geq t^*\,, \tag{20}$$

or equivalently $z(t) + (k \star z)(t) = f(t) + \delta(t)$. We have proven that the solution of (20) is given by

$$z(t) = [f+\delta](t) - (r \star [f+\delta])(t)\,,$$

where r is a convolution operator of type $L^1(\mathbb{R}_{\geq t^*})$, see Theorem 2 and Appendix B. Now, since f is bounded (see Appendix C) and $\delta \in L^\infty(\mathbb{R}_{\geq 0})$ then $[f+\delta] \in L^\infty(\mathbb{R}_{\geq t^*})$. Hence, according to Lemma 8, we can ensure that $z \in L^\infty(\mathbb{R}_{\geq 0})$.

VII. Numerical Example

Consider (4) with the parameters $a_1 = 0.2$, $a_2 = 1$, $c_1 = 0.1$, $c_2 = 0.4$, $b = 1$, $\tau_1 = 0.05$, $\tau_2 = 0.11$, $\bar{\tau}_1 = 0.07$, $\bar{\tau}_2 = 0.09$. The values of these parameters were chosen in the same order as those obtained in [6]. Of course, they satisfy all the conditions of Theorem 1. The simulations were made with Matlab by using an Explicit Euler integration method with a step of $1ms$. In Fig. 1 we can observe the system's state for a simulation with initial condition $x_0 = 0$ in the nominal case. Fig. 2 shows the control signal. In Fig. 3 we can see a simulation considering a disturbance $\delta(t) = \sin(10t)/10$. Note in Fig. 4 that, for this example, the amplitude in steady state is less than the amplitude of the disturbance.

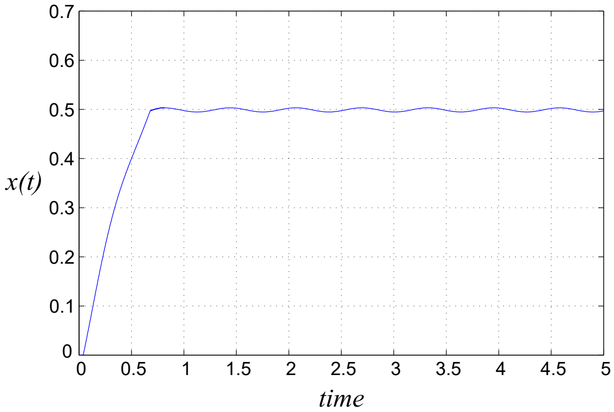

Fig. 3. State of the system in presence of disturbance.

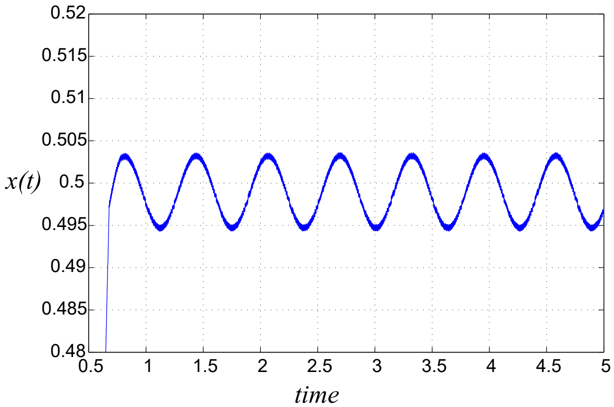

Fig. 4. State of the system in presence of disturbance (amplification).

VIII. Conclusions

We proposed a Sliding Mode Controller for a class of scalar bilinear systems with delays. We have shown that the combination of Lyapunov function and Volterra operator theory provides a very useful tool to study the stability and robustness properties of the proposed control scheme. Naturally, a future direction in this research is to try to extend the control scheme to higher order systems.

Appendix

A. Solutions of delayed differential equations

The theory recalled in this section was taken from [2], see also [11] and [9]. Consider the system

$$\dot{x} = \sum_{i=1}^{N} a_i(t)x(t-\tau_i)\,, \tag{21}$$

where each $\tau_i \in \mathbb{R}_{\geq 0}$, and each a_i is a Lebesgue-measurable and locally essentially bounded function.

Definition 1: The function $X(t,s)$ that satisfies, for each $s \geq 0$, the problem

$$\dot{x} = \sum_{i=1}^{N} a_i(t)x(t-\tau_i)\,,$$

$x(t) = 0$ for $t < s$, $x(s) = 1$, is called the fundamental function of (21).

It is assumed that $X(t,s) = 0$ when $0 \leq t < s$. Now consider the system

$$\dot{x} = \sum_{i=1}^{N} a_i(t)x(t-\tau_i) + f(t)\,, \tag{22}$$

with initial conditions $x(t) = 0$ for all $t < 0$ and $x(0) = x_0$ for some $x_0 \in \mathbb{R}$.

Lemma 3: Assume that a_i, τ_i are as above and f is a Lebesgue-measurable locally essentially bounded function, then there exists a unique solution of (22) and it can be written as

$$x(t) = X(t,0)x_0 + \int_0^t X(t,s)f(s)\,\mathrm{d}s\,.$$

B. Volterra equations

Most of the results recalled in this section was taken from [10]. For $z : \mathbb{R} \to \mathbb{R}$ consider the integral equation

$$z(t) + \int_{t^*}^{t} k(t,s)z(s)\,\mathrm{d}s = f(t)\,, \quad t \geq t^*\,. \tag{23}$$

Define the map $t \mapsto \int_{t^*}^{t} k(t,s)z(s)\,\mathrm{d}s$ as $k \star z$. Hence, we rewrite (23) as

$$z(t) + (k \star z)(t) = f(t)\,. \tag{24}$$

A function r is called a resolvent of (24) if $z(t) = f(t) - (r \star f)(t)$. We say that the kernel $k : J \times J \to \mathbb{R}$ is of type L^p on the interval J if $\|k\|_{L^p(J)} < \infty$, where $\|k\|_{L^p(J)} = \sup_{\substack{\|g_1\|_{L^p(J)} \leq 1 \\ \|g_2\|_{L^p(J)} \leq 1}} \int_J \int_J |g_1(t)k(t,s)g_2(s)|^p \,\mathrm{d}s\,\mathrm{d}t$.

The question about the existence and uniqueness of solutions of (24) is answered by the following lemma.

Lemma 4 ([10], Theorem 9-3.6): If k is a kernel of type L^P on J that has a resolvent r of type L^P on J, and if $f \in L^P(J)$, then (24) has a unique solution $z \in L^P(J)$, and such solution is given by $z(t) = f(t) - (r \star f)(t)$.

Now, we have two problems, verify if k is a kernel of type L^P and if it has a resolvent r of type L^P. For the particular case of $p = 1$ we have the following lemma.

Lemma 5 ([10], Proposition 9-2.7): Let $k : J \times J \to \mathbb{R}$ be a measurable kernel. k is of type L^1 on J if and only if $N(k) = \operatorname{ess\,sup}_{s \in J} \int_J |k(t,s)|\,\mathrm{d}t < \infty$. Moreover $N(k) = \|k\|_{L^p(J)}$.

And finally.

Lemma 6 ([10], Corollary 9-3.10): If k is a kernel of type L^P on J and $\|k\|_{L^p(J)} < 1$, then k has a resolvent r of type L^P on J.

Now we can guarantee some asymptotic behaviour of $z(t)$ according to the asymptotic behaviour of $f(t)$ for a Volterra kernel k. Nonetheless, if such a kernel is of convolution kind, i.e. $k(t,s) = k(t-s)$, we can say something else. For the following lemma let us denote the Laplace transform of $k(t)$ as $K(z)$, $z \in \mathbb{C}$.

Lemma 7 ([10], Theorem 2-4.1): Let k be a Volterra kernel of convolution kind and L^1 type on $\mathbb{R}_{\geq 0}$. Then the resolvent r is of type L^1 on $\mathbb{R}_{\geq 0}$ if and only if $\det(\mathrm{I} + \mathrm{K(z)}) \neq 0$ for all $z \in \mathbb{C}$ such that $\mathrm{Re}\{z\} \geq 0$.

To finalise this section we recall the following lemma that is useful for the robustness analysis.

Lemma 8 ([10], Theorem 2-2.2): Let r be a convolution Volterra kernel of type $L^1(R_{\geq 0})$, and let $b \in L^P(R_{\geq 0})$ for some $p \in [1, \infty]$. Then $r \star b \in L^P(R_{\geq 0})$, and

$$\|r \star b\|_{L^P(R_{\geq 0})} \leq \|r\|_{L^1(R_{\geq 0})} \|b\|_{L^p(R_{\geq 0})} .$$

C. Function f is in L^1

Here we verify that f is in L^1. First note that the integral in f restricts to $t^* - \tau_2 \leq s \leq t^*$, therefore, the argument of $k(t-s)$ is restricted to $t - t^* \leq t - s \leq t - t^* + \tau_2$. Recall that $k(t-s)$ is different from zero only in the interval $[\min(\varsigma, \tau_1), \tau_2]$. Hence, under the integral in f, $k(t-s)$ can be different from zero only for $t^* + \min(\varsigma, \tau_1) - \tau_2 \leq t \leq t^* + \tau_2$. Thus

$$||f(t)||_{L^1(\mathbb{R}_{\geq 0})} = \int_0^\infty |f(t)| \, \mathrm{d}t = \int\limits_{t^*+\min(\varsigma,\tau_1)-\tau_2}^{t^*+\tau_2} |f(t)| \, \mathrm{d}t \, .$$

Now, since $\phi(t) = x(t) - x^*$ and $x(t)$ was guaranteed to be bounded then there exists a finite $\phi^* \in \mathbb{R}_{\geq 0}$ such that $|\phi(t)| \leq \phi^*$ for all $t \in [t^* - \tau_2, t^*]$. Note that also k is bounded by some finite k^*, thus

$$\|f(t)\|_{L^1(\mathbb{R}_{\geq 0})} \leq \int\limits_{t^*+\min(\varsigma,\tau_1)-\tau_2}^{t^*+\tau_2} \left| \int_{t^*-\tau_2}^{t^*} k^* \phi^* \, \mathrm{d}s \right| \mathrm{d}t \, ,$$

therefore

$$\|f(t)\|_{L^1(\mathbb{R}_{\geq 0})} \leq k^* \phi^* \tau_2 (2\tau_2 - \min(\varsigma, \tau_1)) \, .$$

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